Exact solutions to the Mo-Papas and Landau-Lifshitz equations

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Two exact solutions of the Mo-Papas and Landau-Lifshitz equations for a point charge in classical electrodynamics are presented here. Both equations admit as an exact solution the motion of a charge rotating with constant speed in a circular orbit. These equations also admit as an exact solution the motion of two identical charges rotating with constant speed at the opposite ends of a diameter. These exact solutions allow one to obtain, starting from the equation of motion, a definite formula for the rate of radiation. In both cases the rate of radiation can also be obtained, with independence of the equation of motion, from the well known fields of a point charge, that is, from the Maxwell equations. The rate of radiation obtained from the Mo-Papas equation in the one-charge case coincides with the rate of radiation that comes from the Maxwell equations; but in the two-charge case the results do not coincide. On the other hand, the rate of radiation obtained from the Landau-Lifshitz equation differs from the one that follows from the Maxwell equations in both the one-charge and two-charge cases. This last result does not support a recent statement by Rohrlich in favor of considering the Landau-Lifshitz equation as the correct and exact equation of motion for a point charge in classical electrodynamics.

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I. INTRODUCTION

According to the Maxwell equations a point charge in motion emits, in general, radiation that carries away energy and momentum, and that in consequence has an influence on the trajectory of the charge. Although this physical picture is rather obvious, its incorporation in the construction of a consistent equation of motion has been a matter of long debate that still remains nowadays. What lies behind this difficulty is the fact that the construction of an equation of motion for a point charge that takes into account the charge's self-field is intimately linked with fundamental problems, such as the infinite self-energy of a point charge and mass renormalization, which have not been solved yet in a satisfactory way in either classical electrodynamics or quantum electrodynamics.

The most prominent equation of motion for a point charge in classical electrodynamics is the Lorentz-Dirac equation [1,2]. This equation exhibits runaway solutions, where the charge accelerates even in the absence of an external field. These pathological solutions can be removed by imposing the so called asymptotic condition, in which the acceleration is required to vanish after a long time. But this asymptotic condition has as an unavoidable mathematical consequence the existence of acausality or preacceleration, where the charge begins to accelerate before the force is applied. This violation of causality involves very tiny intervals of time, of the order of 10^{-23} s in the case of the electron, and therefore it cannot have any measurable effects in experiments where classical electrodynamics is applicable. Preacceleration is, nevertheless, considered unacceptable because it is incompatible with the principle of causality, which is assumed to be valid for any time interval, no matter how small it may be.

In order to avoid the embarrassing features of runaway solutions and acausality that affect the Lorentz-Dirac equation, several alternative equations of motion have been proposed [3–8]. But, as has been noticed by Parrott [8], and in contrast with the Lorentz-Dirac equation, these alternative equations of motion have not been subjected to detailed scrutiny. Notwithstanding, Huschilt and Baylis [9] found evidence against the general validity of the Mo-Papas equation, and Comay [10] showed that the Mo-Papas [5], Bonnor [6], and Herrera [7] equations are incompatible with the principle of energy conservation.

All the alternative equations of motion [3–8] are based on the idea that in experiments where classical electrodynamics is applicable the effect of the radiation constitutes a small correction to the acceleration, the latter being determined by the equation that completely neglects the radiation reaction force. Thus, it seems almost impossible to discriminate between the different equations of motion by means of experiments. Shen [11], for example, found that, for a charge moving in a constant magnetic field, the Mo-Papas equation [5] gives the same observable results as the Lorentz-Dirac equation. Therefore, the search for a correct equation of motion must be carried out on theoretical grounds, that is, by examining from a mathematical point of view the consistence of a given equation of motion with basic principles like causality, energy conservation and the natural requirement that the rate of radiation associated with an equation of motion for a point charge must coincide with the rate of radiation coming from the fields of a point charge; fields that in turn are determined by means of the Maxwell equations.

An alternative equation of motion that has gained renewed interest recently is the Landau-Lifshitz equation. In particular, Rohrlich [12] considers that the Landau-Lifshitz equation is an exact equation of motion, and the correct one, for a point charge in classical electrodynamics. However, we will show below that the Landau-Lifshitz equation leads to an incorrect rate of radiation for a point charge.

We will study in this paper the rate of radiation associated with the Mo-Papas and Landau-Lifshitz equations, by means of the construction of exact solutions of these equations. The Mo-Papas and the Landau-Lifshitz equations determine the world line of a particle that is characterized only by its mass and charge; therefore both equations are equations of motion for a point charge. Now, for some particular solutions of a given equation of motion it is possible to obtain, starting from the solution, a clear and conclusive result for the rate of radiation associated with that motion. On the other hand, the rate of radiation can also be obtained starting from the well known fields of a point charge, by calculating the flux of the Poynting vector across a sphere of very large radius that encloses the charges.

As we will show here, the Mo-Papas and Landau-Lifshitz equations admit as exact solutions, with appropriate external fields, the motion of a charge in a circular orbit with constant speed, as well as the motion of two identical charges that rotate with constant angular velocity at the opposite ends of a diameter. For each solution the Mo-Papas and the Landau-Lifshitz equations each lead to an exact analytical formula for the rate of radiation, which can be compared with the calculation based on the Poynting vector.

In contrast to the case of the motion along a straight line, where the Mo-Papas equation does not consider any radiation at all [9], in the case of a charge in circular orbit with constant speed the rate of radiation associated with the Mo-Papas equation is exactly the same as the rate of radiation that follows from the fields of a point charge. However, in the case of the two identical charges, the Mo-Papas equation fails to correctly describe the part of the rate of radiation that corresponds to the interference of the fields of the two charges. On the other hand, in the case of the Landau-Lifshitz equation, both solutions lead to a rate of radiation that does not coincide with the one obtained from the fields of a point charge.

II. THE EXTERNAL FIELDS

The construction of the exact solutions of this paper requires the assistance of external electromagnetic fields. They consist of a uniform, time-independent magnetic field, together with a time-independent electric field that is tangent to a family of concentric circumferences contained in planes orthogonal to the magnetic field, and such that the electric field has a fixed magnitude over each circumference. The idealized source that gives rise to the uniform magnetic field is well known, and consists of a flat unbounded current sheet, in which a current flows with constant density and direction everywhere [13]. The idealized source for the external electric field is not so well known, and consists of an infinitely long solenoid of radius b, less than the orbit radius a, and whose axis coincides with the z axis, and that carries a density of current $\boldsymbol{J}$ given by

$$\boldsymbol{J}(\rho,t)=At\,\delta(\rho-b)\hat{\boldsymbol{\varphi}}, \tag{2.1}$$

where A is a positive number, δ is the usual Dirac delta function, ρ is the radial cylindrical coordinate, and $\hat{\boldsymbol{\varphi}}$ denotes the unit vector associated with the cylindrical coordinate φ. Moreover, Gaussian units will be used throughout this work. The Maxwell equations for the current density (2.1) and a density of charge that vanishes everywhere can be solved exactly [14], and the solution for $\rho>b$ is such that the magnetic field is identically zero, while the electric field is given by

$$\boldsymbol{E}(\rho,\varphi,z)=-\frac{2\pi A b^2}{c^2\rho}\hat{\boldsymbol{\varphi}}. \tag{2.2}$$

The magnitude of this electric field can be adjusted to any desired value by properly choosing the parameter A in Eq. (2.1).

Any idealized source can be, of course, realized experimentally only in an approximate way. However, we emphasize that this paper deals only with fundamental theoretical aspects, that is, with mathematical properties of the equation of motion, and therefore experimental or practical aspects are not pertinent here. In this context the relevance of the idealized sources is that they generate electromagnetic fields with a simple mathematical structure, which in turn allows the construction of exact solutions to the Lorentz-Dirac [14] as well as to the Mo-Papas and Landau-Lifshitz equations.

III. THE MO-PAPAS EQUATION

The Mo-Papas equation is the following one [5]:

$$\dot{v}^{\mu}=(e/mc)F^{\mu\lambda}v_{\lambda}+\frac{2e^3}{3m^2c^4}(F^{\mu\lambda}\dot{v}_{\lambda}-F^{\lambda\alpha}\dot{v}_{\lambda}v_{\alpha}v^{\mu}/c^2), \tag{3.1}$$

where e and m denote the charge and mass of the particle, respectively, c is the velocity of light, $F^{\mu\nu}$ is the field-strength tensor, which contains the fields that act on the charge, v^{μ} is the four-velocity, and the overdot on v^{μ} denotes the derivative with respect to the invariant proper time τ. The metric in Eq. (3.1) is diag$(-1,+1,+1,+1)$.

We will now apply Eq. (3.1) to the motion of a charge that rotates with constant speed in a circular orbit that lies in the x-y plane, has its center at the origin of the coordinate system, and has a radius a. In other words,

$$x^0=ct,\quad x^1=a\cos\omega t,\quad x^2=a\sin\omega t,\quad x^3=0, \tag{3.2}$$

where ω is a time-independent parameter. In addition, it will be assumed that the field-strength tensor $F^{\mu\nu}$ has only the following nonvanishing components:

$$F^{01}=-F^{10}=E_x,\quad F^{02}=-F^{20}=E_y,\quad F^{12}=-F^{21}=B_z. \tag{3.3}$$

Thus, the electric field $\boldsymbol{E}$ that acts on the charge does not have a component along the z axis, and the magnetic field $\boldsymbol{B}$ points along the z axis. In this case the component $\mu=3$ of

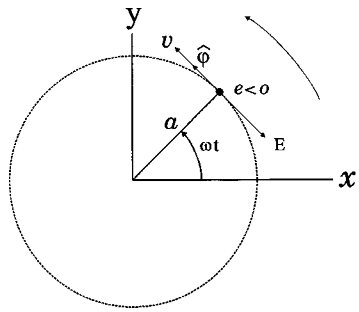

FIG. 1. Circular orbit of radius a of the charge $e<0$. The charge moves in the counterclockwise direction with angular velocity ω and speed $v=\omega a$. The external electric field **E** compensates the radiation reaction force. We also show the tangential unit vector $\hat{\boldsymbol{\varphi}}$ at the electron position.

Eq. (3.1) does not impose any restriction either on the different quantities that characterize the motion, or on the electric and magnetic fields given by Eq. (3.3). The component $\mu=1$ of Eq. (3.1) is

$$\cos\omega t=-(ae/mc^2\beta^2\gamma)(E_x+\beta B_z\cos\omega t)+(2e^3\gamma^2/3m^2c^4)\times(B_z+\beta E_x\cos\omega t+\beta E_y\sin\omega t)\sin\omega t, \tag{3.4}$$

and the component $\mu=2$ is

$$\sin\omega t=-(ae/mc^2\beta^2\gamma)(E_y+\beta B_z\sin\omega t)-(2e^3\gamma^2/3m^2c^4)\times(B_z+\beta E_x\cos\omega t+\beta E_y\sin\omega t)\cos\omega t, \tag{3.5}$$

where the parameter β denotes the quantity $a\omega/c$. To obtain Eqs. (3.4) and (3.5), the proper time derivatives that appear in Eq. (3.1) have been transformed into derivatives with respect to the ordinary time t by means of the relation

$$dt/d\tau=\gamma=(1-\beta^2)^{-1/2}. \tag{3.6}$$

The component $\mu=0$ of Eq. (3.1) can also be obtained starting from Eqs. (3.4) and (3.5). So, in what follows, it will be ignored. Equations (3.4) and (3.5) can be cast in a more suitable form as follows:

$$E_x\cos\omega t+E_y\sin\omega t=-\beta B_z-\frac{mc^2\beta^2\gamma}{ae}, \tag{3.7}$$

$$E_x\sin\omega t-E_y\cos\omega t=\frac{2e^2\beta^2\gamma}{3mc^2a}B_z-\frac{2e\beta^5\gamma^4}{3a^2}. \tag{3.8}$$

IV. THE EXACT SOLUTION OF THE MO-PAPAS EQUATION FOR ONE CHARGE

In this section we will show that Eqs. (3.7) and (3.8) admit as an exact solution the motion of a charge that rotates with constant speed in a circular orbit. To this end we will assume the existence of an external, uniform, time-independent magnetic field $\boldsymbol{B}=B\hat{\boldsymbol{k}}$, along with the external electric field $\boldsymbol{E}$ given by Eq. (2.2). We will also assume that the particle has a negative charge e; therefore, the charge will move in a counterclockwise direction, as shown in Fig. 1.

According to Fig. 1, the components of the external electric field acting on the charge are

$$E_x=E\sin\omega t,\quad E_y=-E\cos\omega t. \tag{4.1}$$

Introducing these quantities and the uniform, time-independent magnetic field $\boldsymbol{B}=B\hat{\boldsymbol{k}}$ in Eqs. (3.7) and (3.8), it is immediately seen that they have the unique solution

$$B=-\frac{mc^2\beta\gamma}{ae}, \tag{4.2}$$

$$E=-\frac{2e\beta^3\gamma^4}{3a^2}. \tag{4.3}$$

Due to the symmetries associated with the motion represented in Fig. 1, in this case it is easy to get the rate of radiation that is implicit in the Mo-Papas equation (3.1). In fact, Eq. (4.3) can be written as follows:

$$e\boldsymbol{v}\cdot\boldsymbol{E}=\frac{2e^2c}{3a^2}\beta^4\gamma^4. \tag{4.4}$$

The quantity $e\boldsymbol{v}\cdot\boldsymbol{E}$ represents the power that the external electric field supplies to the charge. In addition, in this case the kinetic energy of the charge remains constant; and, because of the rotational symmetry, the positions of the charge at different times are indistinguishable. Therefore, assuming the conservation of the energy, all the power supplied by the external electric field to the charge must be necessarily radiated away. In other words, the right hand side of Eq. (4.4) represents, according to the Mo-Papas equation, the rate of radiation for a charge in circular motion. This result coincides with the one obtained by calculating the energy flux across the surface of a sphere of very large radius, using the fields of a point charge that rotates at constant angular velocity in a circle of radius a [15].

V. AN EXACT SOLUTION OF THE MO-PAPAS EQUATION FOR TWO CHARGES

In this section we will show that, for properly chosen external fields of the kind used in the previous section, the Mo-Papas equation (3.1) also admits as an exact solution the motion of two identical charges that rotate with constant angular velocity at the opposite ends of a diameter. Of course, in this case the motion of one of the charges is influenced also by the fields of the other charge. The electric field $\boldsymbol{E}_{ret}$ generated by a point charge at a point $\boldsymbol{x}$ and time t is given by the well known formula

$$\boldsymbol{E}_{ret}=e\left[\frac{(\hat{\boldsymbol{n}}-\boldsymbol{\beta})(1-\beta^2)}{s^3R^2}\right]+\frac{e}{c}\left[\frac{\hat{\boldsymbol{n}}\times\{(\hat{\boldsymbol{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{s^3R}\right], \tag{5.1}$$

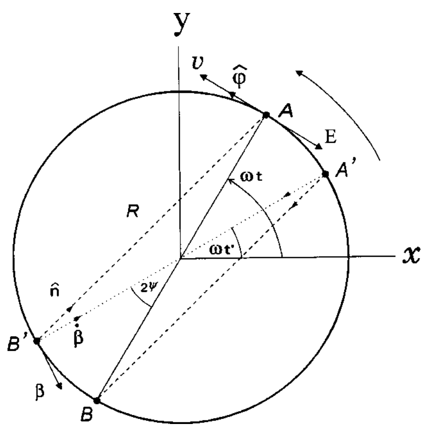

FIG. 2. Two identical charges rotating at constant angular velocity ω in a circular orbit centered at the origin and contained in the x-y plane. At the observation time t the charges are located at points A and B, while at the retarded time t' they are located at points A' and B'.

where $\hat{\boldsymbol{n}}$ is the unit vector that points from the retarded position $\boldsymbol{r}(t')$ of the charge to the point $\boldsymbol{x}$; R is the retarded distance defined by $R=|\boldsymbol{x}-\boldsymbol{r}(t')|$; $\boldsymbol{\beta}$ and $\dot{\boldsymbol{\beta}}$ are defined by $(1/c)d\boldsymbol{r}'/dt'$ and $d\boldsymbol{\beta}/dt'$, respectively, and both are evaluated at the retarded time t', which is implicitly defined by $t=t'+R/c$; and s is the positive number

$$s=1-\hat{\boldsymbol{n}}\cdot\boldsymbol{\beta}. \tag{5.2}$$

The magnetic field $\boldsymbol{B}_{ret}$ is given by

$$\boldsymbol{B}_{ret}=\hat{\boldsymbol{n}}\times\boldsymbol{E}_{ret}. \tag{5.3}$$

The motion of the two charges takes place in the x-y plane, as shown in Fig. 2.

In what follows we will apply the Mo-Papas equation (3.1) to the charge located in the first quadrant of Fig. 2, whose motion is described by Eq. (3.2). Figure 2 also shows the position of the charges at the retarded time t', which is such that a light signal takes the time $t-t'$ to go from the retarded position B' of the second charge to the present position A of the first charge where the retarded fields (5.1) and (5.3) are evaluated. If the angle between the diameter A-B and the diameter A'-B' is denoted by 2ψ, then from Fig. 2 it is immediately seen that the angle ψ is related to the parameter $\beta=a\omega/c$ by means of

$$\psi=\beta\cos\psi. \tag{5.4}$$

Now, since $d\beta/d\psi>0$, this equation implies a one-to-one relationship between β and the angle ψ. Although the parameters β and ψ are not independent, in what follows we will use both of them in the formulas.

It follows from Fig. 2 that the retarded electric field (5.1), evaluated at the position of the charge in the first quadrant, has a vanishing component along the z axis, and that the corresponding magnetic field (5.3) has a vanishing component parallel to the x-y plane. Therefore, the form of the Mo-Papas equation given by Eqs. (3.7) and (3.8) remains valid for the motion shown in Fig. 2.

Figure 2 shows that

$$R=2a\cos\psi; \tag{5.5}$$

while Eq. (5.2) becomes

$$s=1+\beta\sin\psi. \tag{5.6}$$

Now, the components of the retarded electric field $E_{ret\,x}$ and $E_{ret\,y}$ acting over the charge are

$$E_{ret\,x}=(e/\gamma^2R^2s^3)\{\cos(\omega t-\psi)-\beta\sin(\omega t-2\psi)\} -(e\beta^2/aRs^3)(\beta+\sin\psi)\sin(\omega t-\psi), \tag{5.7}$$

$$E_{ret\,y}=(e/\gamma^2R^2s^3)\{\sin(\omega t-\psi)+\beta\cos(\omega t-2\psi)\} +(e\beta^2/aRs^3)(\beta+\sin\psi)\cos(\omega t-\psi), \tag{5.8}$$

while the retarded component of the magnetic field $B_{ret\,z}$ is given by

$$B_{ret\,z}=(e\beta/\gamma^2R^2s^3)\cos\psi+(e\beta^2/aRs^3)(\beta+\sin\psi). \tag{5.9}$$

Because of the superposition principle, the components of the electric and magnetic fields that appear in the Mo-Papas Eqs. (3.7) and (3.8) are the following:

$$E_x=E\sin\omega t+E_{ret\,x},$$
$$E_y=-E\cos\omega t+E_{ret\,y}, \tag{5.10}$$
$$B_z=B+B_{ret\,z},$$

where the magnitude of the external electric field is denoted by E, while the magnitude of the external magnetic field is denoted by B.

It is easy to show that the Mo-Papas equations (3.7) and (3.8) are satisfied in this case by the following values of B and E:

$$B=-\frac{mc^2\beta\gamma}{ae}-\frac{1+\beta\sin\psi}{\beta}E_{ret\,\rho}-\cos\psi E_{ret\,\varphi}, \tag{5.11}$$

$$E=-\frac{2e}{3a^2}\beta^3\gamma^4+E_{ret\,\varphi}-\beta\gamma\lambda E_{ret\,\rho}, \tag{5.12}$$

where $E_{ret\,\varphi}$ is the component of Eq. (5.1) along the unit tangent vector $\hat{\boldsymbol{\varphi}}$ of Fig. 2, that is,

$$E_{ret\,\varphi}=\boldsymbol{E}_{ret}\cdot\hat{\boldsymbol{\varphi}}=\left(\frac{e}{4a^2}\right)\left\{\frac{2\beta}{(1+\beta\sin\psi)^2}-\frac{(1-\beta^2)(\beta+\sin\psi)}{\cos^2\psi(1+\beta\sin\psi)^3}\right\}, \tag{5.13}$$

and $E_{ret\ \rho}$ is the component of (5.1) along the unit radial vector $\hat{\boldsymbol{\rho}}=\cos\omega t\,\hat{\boldsymbol{i}}+\sin\omega t\,\hat{\boldsymbol{j}}$, and is given by

$$E_{ret\ \rho}=\boldsymbol{E}_{ret}\cdot\hat{\boldsymbol{\rho}}=\left(\frac{e}{4a^2}\right)\left\{\frac{2\beta\sin\psi}{\cos\psi(1+\beta\sin\psi)^2}+\frac{(1-\beta^2)}{\cos\psi(1+\beta\sin\psi)^3}\right\}. \tag{5.14}$$

The dimensionless parameter λ that appears in Eq. (5.12) is given by

$$\lambda=\frac{2e^2}{3mc^2a} \tag{5.15}$$

and corresponds to 2/3 of the ratio of the classical electron radius to the orbit's radius. It is therefore a very small number for any macroscopic motion.

When the Mo-Papas equation is applied to the second charge, that is, to the charge that at time t is located in the third quadrant of Fig. 2, Eqs. (5.11) and (5.12) are obtained again. This is, of course, an expected result on account of the symmetries of the motion. In other words, the equation of Mo and Papas admits as an exact solution the motion of two identical charges rotating at constant angular velocity at the opposite ends of a diameter. The magnitudes of the external magnetic field and of the external electric field that make possible this motion are given by Eqs. (5.11) and (5.12), respectively. The effect of the second charge on the motion of the first charge is described by the terms that involve $E_{ret\ \varphi}$ and $E_{ret\ \rho}$ in Eqs. (5.11) and (5.12). The solution reduces, of course, to the solution for one charge when the retarded effects are thrown away.

In order to obtain the rate of radiation for the system of two charges shown in Fig. 2, we note that $E_{ret\ \varphi}$ is negative [14], and since $e<0$, Eq. (5.12) can be written in the form

$$2e\boldsymbol{v}\cdot\boldsymbol{E}=\frac{2e^2c}{3a^2}\beta^4\gamma^4+\frac{2e^2c}{3a^2}\beta^4\gamma^4-2e\boldsymbol{v}\cdot\boldsymbol{E}_{ret\ \varphi}+\lambda(2ec\beta^2\gamma E_{ret\ \rho}). \tag{5.16}$$

By the same discussion that follows Eq. (4.4), the right hand side of Eq. (5.16) represents, according to the Mo-Papas equation, the rate of radiation for the system of two charges shown in Fig. 2. But the rate of radiation can also be determined with the help of the Liénard-Wiechert fields for a point charge, Eqs. (5.1) and (5.3), by computing, for the motion of Fig. 2, the energy flux across a spherical surface of a very large radius that encloses the charges. It turns out that this procedure leads, by means of an analytical and rigorous calculation [16], to the result that the rate of radiation is exactly given by only the first three terms of the right hand side of Eq. (5.16). The fact that the total rate of radiation given by the right hand side of Eq. (5.16) is incorrect can also be seen from the fact that, for a given motion of the charges, the rate of radiation has no relation to the mass of the charges. However, the mass of the charges appears in Eq. (5.16) through the parameter λ. From Eq. (5.16) it is evident that the Mo-Papas equation does not describe correctly the part of the rate of radiation associated with the interference of the fields of the two charges shown in Fig. 2.

VI. THE LANDAU-LIFSHITZ EQUATION

Using the same notation as in Sec. II, the Landau-Lifshitz equation is [4]

$$\dot{v}^{\mu}=(e/mc)F^{\mu\lambda}v_{\lambda}+\frac{2e^3}{3m^2c^4}[\dot{F}^{\mu\lambda}v_{\lambda}+(e/mc)F^{\mu\lambda}F_{\lambda\rho}v^{\rho}+(e/mc)F^{\lambda\alpha}F_{\alpha\rho}v^{\rho}v_{\lambda}v^{\mu}/c^2], \tag{6.1}$$

where the overdot on $F^{\mu\lambda}$ denotes its derivative with respect to the proper time τ. When Eqs. (3.2) and (3.3) are imposed, the component $\mu=3$ of Eq. (6.1) is identically satisfied, while the component $\mu=1$ becomes

$$\cos\omega t=-(ae/\gamma mc^2\beta^2)(E_x+\beta B_z\cos\omega t)-(2e^3a/3m^2c^5\beta^2\gamma)\{\gamma(dE_x/dt+\beta dB_z/dt\cos\omega t)+(e/mc)[\beta E_xN+B_z(E_y+\beta\sin\omega tB_z)+\beta\sin\omega t(N^2+\gamma^2M^2+2\beta\gamma^2B_zM+\beta^2\gamma^2B_z^2)]\}, \tag{6.2}$$

where the derivatives with respect to the proper time τ have been transformed into derivatives with respect to the laboratory time t by means of the relation $dt/d\tau=\gamma$, and where the following notation has been introduced:

$$M=E_x\cos\omega t+E_y\sin\omega t,\qquad N=-E_x\sin\omega t+E_y\cos\omega t. \tag{6.3}$$

Similarly, the component $\mu=2$ of Eq. (6.1) becomes

$$\sin\omega t = -(ae/\gamma mc^2\beta^2)(E_y+\beta B_z \sin\omega t)-(2e^3a/3m^2c^5\beta^2\gamma)\{\gamma(dE_y/dt+\beta dB_z/dt\sin\omega t)+(e/mc)[\beta E_y N - B_z(E_x + \beta\cos\omega t B_z)-\beta\cos\omega t(N^2+\gamma^2M^2+2\beta\gamma^2B_zM+\beta^2\gamma^2B_z^2)]\}. \tag{6.4}$$

The component $\mu=0$ of Eq. (6.1) can be obtained as a combination of Eqs. (6.2) and (6.4) and therefore it will not receive further consideration.

A. The exact solution for one charge

Equations (6.2) and (6.4) admit as an exact solution the motion of a charge in a circular orbit in the presence of the external, uniform, time-independent magnetic field $\boldsymbol{B}=B\hat{\boldsymbol{k}}$ and of the external electric field given by Eq. (2.2). The expressions for the Cartesian components of the external fields

$$E_x=E\sin\omega t,\quad E_y=-E\cos\omega t,\quad B_z=B \tag{6.5}$$

satisfy the equations

$$\sin\omega t\, dE_x/dt-\cos\omega t\, dE_y/dt=0,$$
$$\cos\omega t\, dE_x/dt+\sin\omega t\, dE_y/dt=\omega E, \tag{6.6}$$
$$M=0,\ \ N=-E,$$

and therefore Eqs. (6.2) and (6.4) can be combined into the equations

$$E=-(2e^3/3m^2c^4)\beta\gamma^2B^2, \tag{6.7}$$

$$\gamma\beta^2c/a+\beta(eB/mc)+(2e^3/3m^2c^4)E\{\gamma\beta c/a-(eB/mc)\}=0. \tag{6.8}$$

These equations, however, cannot be satisfied if the magnitude of the external electric field is given by $E=-(2e\beta^3\gamma^4/3a^2)$, that is, by the external field that leads to the correct rate of radiation, namely, $(2e^2c/3a^2)\beta^4\gamma^4$ for a charge performing the motion shown in Fig. 1. Thus, according to the discussion that follows Eq. (4.4), the Landau-Lifshitz equation predicts an incorrect rate of radiation.

Introducing an auxiliary variable α according to $E=-(2e/3a^2)\beta^3\gamma^4\alpha^2$, Eqs. (6.7) and (6.8) combine into a cubic equation for α:

$$\lambda^2\beta^2\gamma^4\alpha^3-\lambda^2\beta^2\gamma^4\alpha^2+\alpha+1=0. \tag{6.9}$$

The solutions of the cubic equation are well known [17]. The condition for the existence of a real solution of Eq. (6.9) is

$$\frac{1+11\lambda^2\beta^2\gamma^4-\lambda^4\beta^4\gamma^8}{27\lambda^6\beta^6\gamma^{12}}>0, \tag{6.10}$$

a condition that is satisfied since $\lambda\beta\gamma^2\ll 1$ for any macroscopic motion if γ is not too high. If the real solution of Eq. (6.9) is expressed as a power series in parameter λ, the following result is obtained for the magnitude of the external electric field:

$$E=-(2e\beta^3\gamma^4/3a^2)\times\{1-4\beta^2\gamma^4\lambda^2+24\beta^4\gamma^8\lambda^4+\cdots\}, \tag{6.11}$$

which shows explicitly that the departure of the Landau-Lifshitz equation from the fields of a point charge is a small one. Similarly, the external magnetic field is given by

$$B=-(mc^2\beta\gamma/ea)\times\{1-2\beta^2\gamma^4\lambda^2+10\beta^4\gamma^8\lambda^4+\cdots\}. \tag{6.12}$$

B. The exact solution of the Landau-Lifshitz equation for two identical charges

The motion shown in Fig. 2 can also be obtained as an exact solution of the Landau-Lifshitz equation. The components of the electric and magnetic fields are given by

$$E_x=E\sin\omega t+E_{ret\,x},$$
$$E_y=-E\cos\omega t+E_{ret\,y}, \tag{6.13}$$
$$B_z=B+B_{ret\,z},$$

where $E_{ret\,x}$, $E_{ret\,y}$ and $B_{ret\,z}$ are given by Eqs. (5.7), (5.8), and (5.9). From here it follows that

$$\sin\omega t\, dE_x/dt-\cos\omega t\, dE_y/dt=-\omega E_{ret\,\rho},$$
$$\cos\omega t\, dE_x/dt+\sin\omega t\, dE_y/dt=-\omega(E_{ret\,\varphi}-E),$$
$$M=E_{ret\,\rho},\quad N=E_{ret\,\varphi}-E, \tag{6.14}$$

where the radial and tangential retarded fields $E_{ret\,\rho}$ and $E_{ret\,\varphi}$ are given by Eqs. (5.14) and (5.13), respectively. Using these results it is possible to cast the Landau-Lifshitz equations (6.2) and (6.4) in a more convenient form. To this end it is useful to introduce dimensionless electric and magnetic fields by explicitly factoring out the factor $(mc^2\beta\gamma/ae)$. Thus,

$$B=(mc^2\beta\gamma/ae)\overline{B}, \tag{6.15}$$

with similar expressions defining $\overline{E}$, $\overline{E}_{ret\,\varphi}$, $\overline{E}_{ret\,\rho}$, and $\overline{B}_{ret\,z}$. In terms of these quantities, the Landau-Lifshitz equations (6.2) and (6.4) can be cast in the following form:

$$\overline{E}-\overline{E}_{ret\,\varphi}=\lambda\gamma\beta\{\overline{E}_{ret\,\rho}(1-\overline{B}_z)-\beta\gamma^2(\overline{E}^2_{ret\,\rho}+2\beta\overline{E}_{ret\,\rho}\overline{B}_z+\overline{B}^2_z)\}, \tag{6.16}$$

$$\beta+\overline{E}_{ret\ \rho}+\beta\overline{B}_z +\lambda\beta\gamma(\overline{E}-\overline{E}_{ret\ \varphi})(1-\beta\overline{E}_{ret\ \rho}-\overline{B}_z)=0. \tag{6.17}$$

After substitution of Eq. (6.16) in the right hand side of Eq. (6.17) a cubic equation for $\overline{B}_z$ is obtained. The real solution of this equation determines the external magnetic field B. Also, when this solution is inserted in the right hand side of Eq. (6.16), an expression for the external electric field is obtained. The results, expressed as power series in the parameter λ of Eq. (5.15), are

$$E=-(2e\beta^3\gamma^4/3a^2)\{1-4\beta^2\gamma^4\lambda^2+\cdots\}+E_{ret\ \varphi} +4\beta^3\gamma^5 E_{ret\ \rho}\lambda^3\left\{1+\frac{\gamma\beta}{4}[E_{ret\ \rho}/(2e\beta^3\gamma^4/3a^2)]\lambda +\cdots\right\}, \tag{6.18}$$

$$B=-(mc^2\beta\gamma/ae)\{1-2\beta^2\gamma^4\lambda^2+\cdots\} -(1+\beta\sin\psi)\beta^{-1}E_{ret\ \rho}-\cos\psi E_{ret\ \varphi}+\beta\gamma^2 E_{ret\ \rho}\lambda^2 +\cdots. \tag{6.19}$$

Formula (6.18) shows that the discrepancies between the Landau-Lifshitz equation and the rate of radiation calculated starting from the fields of a point charge are contained not only in the term that describes the rate of radiation of only one charge, given by Eq. (6.11); they are also contained in the term that describes the interference between the fields of the charges.

VII. COMMENTS

The Mo-Papas and Landau-Lifshitz equations do not have runaway solutions or preacceleration effects, as is the case with the Lorentz-Dirac equation. However, as has been clearly shown here, these equations lead to an incorrect rate of radiation. It is interesting to point out in this context that the Lorentz-Dirac equation also admits the motions of Fig. 1 and 2 as exact solutions [14], but in contrast to the Mo-Papas and the Landau-Lifshitz equations, the rate of radiation associated with these motions is the same as the rate of radiation that follows from the fields of a point charge. In a certain sense, it can be said that the Mo-Papas and the Landau-Lifshitz equations solve the well known pathologies of the Lorentz-Dirac equation, but at the price of introducing an incorrect description of the rate of radiation. This departure from the correct rate of radiation is very small, as is the violation of causality in the Lorentz-Dirac equation, since the former is proportional to the parameter λ of Eq. (5.15), which is a very small number for a macroscopic motion. However, from a fundamental point of view, an incorrect description of the rate of radiation is a defect as unsatisfactory as the existence of preacceleration. For this reason, in our opinion, the Landau-Lifshitz equation cannot be considered as a definitive answer in the quest for a correct equation of motion for a point charge in classical electrodynamics [12]. The existence of the above difficulties does not mean, however, that it is impossible to construct an equation of motion for a point charge free of pathologies.

The simplicity of the external fields has played a crucial role in obtaining the results of this paper. Unfortunately, as is usual when one deals with highly idealized sources, these external fields have an infinite field energy stored in them. Nevertheless, we want to emphasize that this inconvenience does not affect the energy balance, which is the essential step to obtain the rate of radiation.

In connection with the external fields, one may also wonder about the interference effects between the fields of the infinitely long solenoid and the fields of the charges in motion. It is not difficult to show that there is no radiation escaping to infinity as a result of this interference. If we denote by $\boldsymbol{E}_s$ and $\boldsymbol{B}_s$ the electric and magnetic fields of the solenoid, and by $\boldsymbol{E}_e$ and $\boldsymbol{B}_e$ the retarded fields of the charge performing the motion in Fig. 1, then we must study the flux of $\boldsymbol{E}_e\times\boldsymbol{B}_s$ and $\boldsymbol{E}_s\times\boldsymbol{B}_e$ across a very large surface. Let us study first the flux over a cap of the same radius as the solenoid, centered on the solenoid's axis and orthogonal to it, and located at $z\rightarrow\pm\infty$. It is immediately seen that the flux of $\boldsymbol{E}_e\times\boldsymbol{B}_s$ across this cap is zero, since $\boldsymbol{E}_e\times\boldsymbol{B}_s$ does not have any component along the z axis. The flux of $\boldsymbol{E}_s\times\boldsymbol{B}_e$ over this cap is also zero, since the area of the cap is finite and the field $\boldsymbol{B}_e\rightarrow 0$.

To study the flux of the vector $\boldsymbol{E}_s\times\boldsymbol{B}_e$ outside the solenoid it is convenient to choose a closed surface formed by a cylindrical surface concentric with the solenoid and of a very large radius, together with its caps parallel to the x-y plane and located at $z\rightarrow\pm\infty$. Let us consider first the flux across the cap. We start by calculating the flux of $\boldsymbol{E}_s\times\boldsymbol{B}_e$ across the surface comprised between the radii ρ and $\rho+d\rho$. Then, the surface element is proportional to ρ, a dependence that is canceled out by the ρ^{-1} dependence of $\boldsymbol{E}_s$. Therefore, the flux across this surface is, roughly speaking, proportional to the field $\boldsymbol{B}_e$, which tends to zero when the caps are located at $z\rightarrow\pm\infty$. In a similar way it can be shown that the flux across the cylindrical surface is zero. Therefore, there is no field energy radiated to infinity due to the interference between the fields of the solenoid and the fields of the charge shown in Fig. 1.

Finally, one may also wonder about the interference effects between the external homogeneous magnetic field and the fields of the charge. In this context we must point out that in the construction of the exact solutions of this paper it is not necessary for the external magnetic field to extend to infinity in the radial direction. In fact, by using an infinitely long solenoid of radius d larger than the orbit radius and fed by a time-independent current density we obtain a constant homogeneous magnetic field that is identically zero for $\rho > d$. In addition, for similar reasons to the ones given in the preceding paragraphs, there is no flux of energy through the caps located at $z\rightarrow\pm\infty$. Therefore, the only field energy that escapes to infinity is the one associated with the fields of the charge.

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